THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; Prof. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
Prof. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
GEORGE BELL & SONS, PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

Vol. V. October, 1909. No. 81.

MEETINGS OF EDUCATIONAL SOCIETIES IN JANUARY, 1910.

Jan. 12.—The Mathematical Association in the morning, and a Joint Meeting of the Mathematical and the Science Masters in the afternoon. The two Associations also meet at dinner in the evening.
Jan. 13.—Public School Science Masters' Association.
Jan. 5-7.—Assistant Masters' Association.
Jan. 8.—Assistant Mistresses' Association.

MEETING OF THE COUNCIL OF THE MATHEMATICAL ASSOCIATION.

A meeting of the Council was held at King's College on October 2nd, 1909.

The deaths of two members since the last meeting were reported —Prof. Simon Newcomb and Mr. C. A. Rumsey, who for some years served on the Council.

There were 19 new members elected, making 68 since the last annual meeting.

Prof. A. H. Leahy and Mr. H. T. Kelsey were appointed to represent the Association on the General Committee of the North of England Education Conference, to be held at Leeds on January 6-8, 1910.

Mr. H. D. Ellis, Hon. Secretary, was appointed a delegate to the British Association of Delegates of Scientific Societies at Burlington House on October 25, 26.

The report of the Sub-Committee on the Formation of Local Branches was considered (v. p. 122). The consideration of

whether steps should be taken towards the formation of a London Branch was left to the Sub-Committee.

A letter from Prof. E. L. Watkin suggesting the affiliation of the Southampton Mathematical Society to the Association was considered.

MATHEMATICAL ASSOCIATION.

THE FORMATION OF LOCAL BRANCHES.

The Council, having considered the question of the formation of local branches, desires to ascertain the opinions of members of the Association on the following proposals which will be laid before the Association at the Annual Meeting on 12th January, 1910. Criticism and suggestions are invited and should be sent to the Secretaries.

PROPOSED REGULATIONS.

1. That local branches of the Association be formed at the discretion of the Council, if desired by local members.

2. That the rules of the Association be altered to allow of the election of Associates, who would be attached to a local branch. The position of Associates would be as follows:

   (*a*) They would receive all reports issued by the Association, but would not receive the *Gazette.*

   (*b*) They would receive intimation of all meetings of the branch to which they are attached, and would participate fully in the management of that branch.

   (*c*) They would receive intimation of all general meetings of the Association, but would not have the right to vote on matters dealing with the financial business of the Association.

   (*d*) The branch would make to the Association a grant of 1/6 for each Associate, and the Secretary of the branch would furnish the names and addresses of such Associates to the Secretaries of the Association.

3. That the Association pay over to a local branch 1s. per annum on behalf of every Member of the Association who has joined that branch, and this should constitute the Member's subscription to the branch.

The Council is prepared to consider the representation of branches on the Council.

At the annual meeting on 12th January, 1910, it will be moved that the rules of the Association be altered in accordance with such recommendations as may then be approved.

SOUTHAMPTON AND DISTRICT MATHEMATICAL SOCIETY.

A meeting was held on Thursday, July 1st, at Hill Crest, Hill Lane, Southampton, to consider the desirability of forming a Society for Southampton and District to facilitate the interchange of ideas amongst those interested in Mathematical Teaching. About a dozen persons were present. The Chair was taken by Prof. Watkin, who said that he had received letters of apology from a number of people who were in sympathy with the movement.

It was resolved that a Mathematical Society should be formed for Southampton and District, whose main object would be the interchange of

ideas on Mathematical Teaching, and after some discussion a provisional committee, consisting of Prof. Watkin (Hartley University College), Mr. C. H. Holmes (S'hampton Grammar School), Mr. Wilde (Taunton School, S'hampton), and Miss E. Hall (S'hampton Girls Grammar School), was appointed to draft a constitution and present it at a meeting to be held early in October. The committee were also requested to report on the question of affiliation with the Mathematical Association.

ANHARMONIC COORDINATES.

(Concluded.)

VIII. Anharmonic Inversion.

(1) (XYZ) and $(X'Y'Z')$ are called *inverts* when $XX'=YY'=ZZ'$, that is when

$$\frac{X}{Y'Z'}=\frac{Y}{Z'X'}=\frac{Z}{X'Y'} \text{ etc.}$$

(2) The line $\quad p\equiv lx+my+nz=0$
and the circumconic $\quad \pi\equiv lyz+mzx+nxy=0$
are inverts.

(3) The meet line of π is

$$p'\equiv\Sigma\frac{x}{l}=0 \text{ [VII. (1)].}$$

The invert of p' is $\quad \pi'\equiv\Sigma\frac{yz}{l}=0.$

The meet line of π' is $\quad p\equiv\Sigma lx=0.$

Hence π and π' are so related that the *meet line* of each is the *inver* of the other.

(4) For the conic $\quad \Sigma yz=0,$

$$\pi=\pi' \text{ and } p=p',$$

i.e. the invert is the meet line.

IX. Anharmonic Reciprocation.

(1) P and p are called reciprocal,
when $\quad P\equiv(X, Y, Z)$
$\quad p\equiv Xx+Yy+Zz=0.$

(2) p is the polar of P for the imaginary conic,

$$x^2+y^2+z^2=0,$$

which reciprocates into itself, to which ABC is self conjugate, and whose centre is (a, b, c), being the pole of the line at infinity, [IV. (4)].
Also $A''B''C''$ is the reciprocal of O.

(3) If the inscribed conic $\theta\equiv\Sigma r^2x^2-2\Sigma styz=0$, then the tangential equation of the reciprocal of θ is

$$\pi\equiv\Sigma r^2l^2-2\Sigma stmn=0.$$

Hence if (xyz) be a point on π, we have

$$\frac{x}{r}=rl-tn-sm, \text{ etc.}$$

Hence the equation of π is

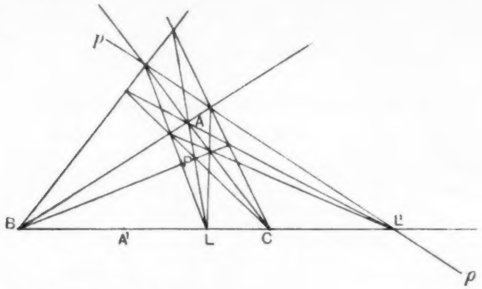
$$\pi\equiv ryz+szx+txy=0 \text{ a circumconic.}$$

(4) Hence, reciprocating VII. (1), we obtain "The joins of the points where θ touches ABC, to the opposite vertices, meet in the point

$$\left(\frac{1}{r}, \frac{1}{s}, \frac{1}{t}\right)."$$

(5) If P and P' lie on p' and p respectively, they are called conjugate points. Condition is $\Sigma XX'=0$.

(6) If p and p' pass through P' and P respectively, they are called conjugate lines. Condition is $\Sigma ll'=0$.



X. (1) *The Harmonic Conjugate.*

Let P be (X, Y, Z).

Let p be the reciprocal of the invert of P, then

$$p\equiv\frac{x}{X}+\frac{y}{Y}+\frac{z}{Z}=0.$$

Then $$-1=\frac{Y}{Z}\cdot-\frac{Z}{Y}=A(BOCP)\div A(BOCL')$$

$$=(BA'CL)\div(BA'CL')=(BL'CL)$$ which is hence harmonic.

With similar results for CA and AB.

Thus P and p may be called harmonic conjugates.

(2) That a harmonic conjugate line exists may be immediately shown by use of Menelaus and Ceva.

(3) $p\equiv\Sigma YZx=0$, may be considered as the polar of P for the cubic $xyz=0$, *i.e.* for the triangle.

Hence P and p may be called pole and polar.

(4) The reciprocals of P and p are

$$\left.\begin{aligned}p_1&\equiv Xx+Yy+Zz=0,\\ P_1&\equiv\left(\frac{1}{X}, \frac{1}{Y}, \frac{1}{Z}\right).\end{aligned}\right\}$$

And these are harmonic conjugates.

Hence $\left.\begin{array}{l}\text{harmonic conjugates point and line reciprocate into}\\ \text{harmonic conjugates line and point.}\end{array}\right\}$

XI. *Geometrical Constructions.*

(1) Given A, B, C, and P, to find p.

Take P as O, then p becomes $A''B''C''$.

Hence construction as in figure to V. (2) or X. (1).

That is, Let AP, BP, CP cut a, b, c in L, M, N

Let LM, MN, NL cut c, a, b in X, Y, Z.

Then XYZ lie in the line p.

(2) Given a, b, c, and p, to find P.

Reciprocate the last construction, we have

Join ap, bp, cp to A, B, C by l, m, n.

Join lm, mn, nl to C, A, B by x, y, z.

Then x, y, z concur at P.

See figure to X. (1).

(3) The following construction is interesting:

Given B, C, P and p, to find A.

Now $$CP \text{ is } \frac{x}{X}=\frac{y}{Y}.$$

Let CP cut p in M. Then

$$BM \equiv \frac{x}{X}+\frac{y}{Y}+\frac{z}{Z}+\left(\frac{x}{X}-\frac{y}{Y}\right)=0,$$

$$i.e. \quad BM \equiv \frac{x}{X}+\frac{z}{2Z}=0.$$

Let BP cut p in L, then $CL \equiv \frac{x}{X}+\frac{y}{2Y}=0.$

Let BM and CL intersect in R, then $AR \equiv \frac{y}{Y}=\frac{z}{Z} \equiv PA$

$\therefore$ P, A, R are collinear.

Let PR cut p in S.

$$\left.\begin{array}{lll} \textit{Then} \text{ by above} & BM \text{ and } CL \text{ intersect in } AP. \\ \text{Similarly} & AM \text{ and } CS \text{ intersect in } BP, \\ \text{and} & AL \text{ and } BS \text{ intersect in } CP. \end{array}\right\}$$

Hence construction:

Let CP, BP cut p in M and L. Let BM, CL cut in R. Let PR cut p in S. Let BS, CS cut CP, BP in U and V. Then PR, UL, MV concur in A.

XI. (4) Reciprocating the last construction, we have:

Given b, c, p and P, to find a. Join cp, bp to P by m and l. Join bm, cl by r. Join pr to P by s. Join bs, cs to cp, bp by u and v. Then pr, ul, mv are collinear on a.

XII. Harmonic Conjugate Envelopes.

$$\left.\begin{array}{l} p \equiv lx+my+nz=0 \\ \pi \equiv lyz+mzx+nxy=0 \end{array}\right\} \text{ are inverts.}$$

Their reciprocals are $$\left.\begin{array}{l} P \equiv (l, m, n) \\ \theta \equiv \Sigma l^2x^2-2\Sigma mnyz=0 \end{array}\right\}.$$

Hence we have

(1) The envelope of the harmonic conjugate lines of all points on a straight line is an inscribed conic.

(2) The conjugate lines of all points on a circumconic pass through a fixed point.

[For the circumconic $\Sigma yz=0$. The fixed point is O.]

And these are reciprocal theorems.

XIII. (1) The relation between

$$\left.\begin{array}{ll} & R_0, \text{ the meet of the harmonic conjugate lines of } P \text{ and } P' \\ \text{and} & R_1, \text{ the harmonic conjugate point of the join } PP'. \end{array}\right\}$$

$$\left.\begin{array}{ll} P(X, Y, Z); & R_0(\xi, \eta, \zeta) \\ P'(X', Y', Z'); & R_1(\xi', \eta', \zeta') \end{array}\right\}.$$

Now $$r \equiv PP' \equiv \|x, y, z'\|=0,$$

$$\therefore\ r \equiv \Sigma(YZ'-ZY')x=0.$$

The reciprocal of r is $(YZ'-ZY', \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots)$.

Whose invert R_1 is $\left(\frac{1}{YZ'-ZY'}, \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots,\right)$.

Also $$p \equiv \Sigma \frac{x}{X}=0;$$

$$\therefore\ R_0 \text{ is given by } \frac{\xi}{\frac{1}{YZ'}-\frac{1}{ZY'}}=\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots$$

$$\therefore\ \frac{\xi}{XX'(YZ'-ZY')}=\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots$$

$$\therefore\ \frac{\xi\xi'}{XX'}=\frac{\eta\eta'}{YY'}=\frac{\zeta\zeta'}{ZZ'}.$$

Hence (1) If P and P' are inverts, so are R_0 and R_1. As is otherwise obvious.

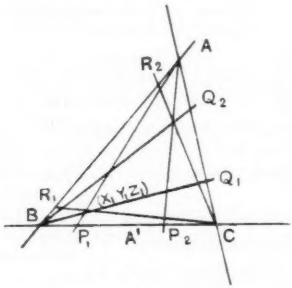
(2) If P and P' are conjugate, so are R_0 and R_1 [by IX. (5)].

XIII. (2) Reciprocating the last theorem.

Relation between

$r_0 \equiv$ the join of the harmonic conjugate points of p_0 and p_1,
and $r_1 \equiv$ the harmonic conjugate line of pp_1.

If p and p_1 are conjugate so are r_0 and r_1.



XIV. (1) Carnot's theorem.

The general conic $\qquad Q\equiv\Sigma lx^2-2\Sigma ryz=0$

cuts BC where $\qquad my^2-2ryz+nz^2=0.$

$$\therefore\ 1=\Pi\frac{n}{m}=\Pi\frac{y_1y_2}{z_1z_2}=\Pi\frac{BA'.CP_1}{A'C.P_1B}\cdot\frac{BA'.CP_2}{A'C.P_2B}$$

$$=\Pi\frac{b^2}{c^2}\cdot\frac{CP_1.CP_2}{BP_1.BP_2}=\Pi\frac{CP_1.CP_2}{BP_1.BP_2}.$$

(2) Hence if AP_1, BQ_1, CR_1 are concurrent, so are AP_2, BP_2, CP_2 (Ceva).

(3) Let Q be the conic determined by $(X_1Y_1Z_1)$ and $(X_2Y_2Z_2)$.

Then
$$\frac{Y^2}{Z^2}-2\frac{r}{m}\cdot\frac{Y}{Z}+\frac{n}{m}=0$$

is equivalent to
$$\frac{Y^2}{Z^2}-\left(\frac{Y_1}{Z_1}+\frac{Y_2}{Z_2}\right)\frac{Y}{Z}+\frac{Y_1Y_2}{Z_1Z_2}=0,$$

that is to
$$\frac{Y^2}{Y_1Y_2}-\left(\frac{1}{Y_1Z_2}+\frac{1}{Z_1Y_2}\right)YZ+\frac{Z^2}{Z_1Z_2}=0.$$

Hence the required conic is

$$\Sigma\frac{X^2}{X_1X_2}-\Sigma\left(\frac{1}{Y_1Z_2}+\frac{1}{Y_2Z_1}\right)YZ=0.$$

TRAINING COLLEGE,
SHEFFIELD.

F. J. O. CODDINGTON.

REVIEWS.

An Introduction to Projective Geometry. By L. N. G. FILON, M.A., D.Sc. Pp. vi, 249. 7s. 6d. 1908. (Arnold.)

Dr. Filon's book, though bearing a similar title to Cremona's work, deals with the subject in an entirely different manner. Whilst in the latter the law of duality is always kept before the notice of the reader, in the former it occupies a more subordinate position, and this allows Dr. Filon to arrange his book more as a continuous treatise, in which the reader has not to stop and, as it were, study each geometrical truth a second time in different words.

Dr. Filon gives us a preliminary chapter on projection, and then in chapter ii. proceeds to the theory of cross-ratio of the range and pencil, and harmonic forms. Considering the importance of this chapter, which is the foundation of the subject and contains within the compass of seventeen pages all that the student is supposed to require before dealing with the conic, it appears too condensed, and several of the examples might with advantage to the beginner have been introduced into the text; *e.g.* all that we are told about two distinct cobasal projective forms is that they are identical if they have more than two self-corresponding elements, but we are not given any help towards finding the latter. No doubt the author was eager to push on to the more attractive part of the subject relating to the conic, and said with Newton, "Propero ad magis utilia." Here Dr. Filon gives us neat proofs of the anharmonic properties of the points and tangents of a conic, followed by a chapter containing the chief properties of poles and polars, which are proved for the circle and then transferred by projection to the conic. We then have the theorems of Pascal and Brianchon, the former proved directly for the conic by cross-ratio, and the latter by reciprocation, and they are employed in the construction of conics subject to different conditions.

These are followed by some elementary propositions which seem somewhat out of place. Although, as Dr. Filon remarks, at present a student has in a way to learn his conics three times over, *i.e.* by the methods of (1) Apollonius, (2) analysis, and (3) projection; on the other hand, considering the importance of the conic in higher geometry, the wisest plan seems to be to reserve the powerful method of projection for the treatment of those advanced parts of the subject for which the other methods are not so suitable. It would hardly be possible to condense into a score of pages an amount of the theory of geometrical conics which would enable a student fully to understand and digest what he might be expected to meet in such a work as the present. Of course it must not be forgotten that projection is only one of the instruments of higher geometry, as Chasles has shown us in his *Traité des Sections Coniques*, where, after deriving in the first half-dozen pages the conic from the circle, the method of projection is laid aside, and the subject is developed entirely by means of the theory of cross-ratio.

After again dealing with Pascal's theorem and constructions depending on it, we are introduced to imaginaries and homography, treated for the most part analytically, the principle of one-one correspondence being carefully explained and its limitations pointed out. After most interesting sections on involution of the line and conic, and homographic forms of the second order, we come to a fascinating chapter on systems of conics, in which the article on the construction of the common self-conjugate triangle is particularly interesting, and makes us ask why Dr. Filon did not go a step further and give us figures showing the pair of real common chords in the different cases to which he refers, viz. when the two conics have only two real intersections, and again when one conic lies entirely inside the other. And here one cannot help saying that the figures do not do the author justice, and are scarcely up to the level of the subject matter. The work concludes with two chapters on the cone and sphere, and quadrics, in which the reader is brought "to the threshold of the rich treasure house of Modern Geometry, and is given a glimpse of some of its more interesting methods."

A considerable amount of new notation is introduced, which, added to a somewhat condensed style, makes it rather difficult to follow the reasoning on a first reading, but the work on the whole is most interesting and suggestive,

and to each chapter is added a moderate number of well-chosen examples. We were disappointed, however, to find that the author has not given any historical notes, which would not have been a difficult matter in the case of a subject of such recent origin. We would point out that the theorem on p. 84, viz. "If through any point two chords are drawn in fixed directions the ratio of the rectangles contained by their segments is the same for all positions of the point," can hardly be called Newton's, seeing that it is employed by Pappus, Bk. VIII., Prop. 13, in his construction of a conic through five points, and is, in effect, merely a re-statement of Apollonius, Bk. III., Prop. 17.

JOHN J. MILNE.

Fragen der Elementargeometrie. By F. ENRIQUES. 1907. (Teubner.)

The mistake is sometimes made of supposing that the success of the movement against the study of the text of Euclid in the schools has been due entirely to the action of the party of which Professor Perry is one of the leaders. Even in England, however, where the needs of the practical man are allowed more weight than abroad, the abandonment of the old routine is due in part to a growing consciousness in mathematical circles that as a logical system Euclid fails to answer to modern philosophical requirements. Outside England the change is of less recent date, and it is entirely to be attributed to the initiative of the mathematicians themselves.

Enriques's collection of essays on *Problems of Elementary Geometry* * has for its object to explain as simply and intelligibly as possible precisely what modern Mathematics has to say in correction, in explanation, and in completion of the old Greek Geometry. It thus indirectly expounds the point of view which outside England led, already many years ago, to the revolution in the study of Geometry above referred to.

The Italian original, which appeared in 1900, consisted of two parts, the first deals with the Axioms of Geometry, the second, translated into German and enlarged, constitutes the volume before us.

The germ of the book is to be found in Klein's *Vorträge über ausgewählte Fragen der Elementargeometrie*" (1895). It may be said at once that the later book constitutes in almost every respect an advance on its prototype. The interest never flags from the first page to the last, although from the fact that various writers are concerned in the authorship and that the topics discussed differ in character, there are, of course, gradations.

The first article on Elementary Methods of solving Geometrical Problems is based on a little book by Professor Petersen of Copenhagen. A translation of this book was published in England under the title *Methods and Theories for the Solution of Problems of Geometrical Constructions*, but the English is so bad that it is in parts barely intelligible. It is perhaps owing to this fact that the book has not had in our country the vogue it has had on the Continent. Petersen's ideas have indeed long been the common property of the school public abroad, and have formed the basis of more than one text-book. The majority of English readers will, however, probably be unacquainted with the contents of this article, which may be said to give an account of almost the only system which has yet been devised for classifying the various methods employed in the solving of geometrical problems.

The second article concerns itself with the possibility of the solution of certain problems by means of the circle alone, no straight lines being drawn. It might be supposed that such an apparently artificial restriction in the use of the appliances at our disposal would have little real interest. This is, however, far from being the case. Both in theory and practice the question discussed is of great importance. The engineer is well aware of the significance of the fact that all straight lines leave the field of operations, while suitably chosen circles do not, and that in machinery circles are much easier to arrive at than straight lines. Apart from these considerations, drawings made with the compasses alone are far more accurate than those involving the use of the ruler. From a theoretical point of view, moreover, there is a

* *Questioni riguardanti la geometria elementare*, Bologna, 1900. German translation by Dr. Fleischer, *Fragen der Elementargeometrie* (Teubner), Leipzig, 1907. Price 9 Marks.

certain charm in these constructions which the writer has succeeded in bringing out.

In contradistinction to the second article, the third deals with the solution of geometrical problems by the use of the ruler alone and without the compasses. It is thus closely connected with Projective Geometry, a subject which is now fairly well known in England, and which, in fact, arose out of such constructions. In the consequential application of this method no measurements are permitted. Subsequently a fixed circle, and towards the end of the article the use of a ruler with two parallel edges, is allowed, as well as that of a T-square, or other angle.

In the fourth article Castelnuovo discusses the question how far it is possible to solve geometrical problems by the use of the various simple instruments used singly and in combination with one another. He shows how this question may be answered by the use of the analytical geometry of Descartes—how, in fact, the problem reduces itself to the solution of certain algebraic equations.

In the fifth article Enriques gives a tolerably complete discussion of the extent to which these algebraic equations are soluble by means of the ruler and compasses. This is equivalent to answering the question, What algebraic equations are soluble by expressions involving square roots only? It leads at once, by a natural transition, not only to the work of Gauss, but also to that of Galois, whose name, however, is not once mentioned in the article except in the title of a work by Bianchi which is cited.*

This fifth article will perhaps be found the most difficult in the volume; it will hardly be regarded as a material simplification of the explanation given by Klein in the book already cited. The feeling that it was desirable to give some more detailed examination of the subject is apparently responsible for the sixth article on the construction of the regular polygon of 17 sides. A clear account is here given of certain constructions chosen by the author for various reasons. Three pages are devoted to an abstract of the analytic solution due to Padoa, which the author cites as particularly worthy of mention on account of its elementary character. The reviewer has quite recently published † an analytical solution which is both simpler and more elementary than that of Padoa. As for the geometrical construction of the polygon of 17 sides, based on the analytic solution of the problem, the best would seem to be that of H. W. Richmond,‡ which, however, is not mentioned in the volume before us, though it is conspicuous for its compactness.

After a complete discussion of the limitations imposed by the use of the simple instruments, illustrated by the application in article VI. of the possible methods of solution, the book passes naturally to the discussion of certain problems interesting in themselves as well as on historical grounds, which are insoluble by means of the ruler and compasses alone. Such problems fall into two classes: what may be called *algebraic problems*, or problems which correspond to algebraic equations, and *problems of a transcendental nature*, that is, problems which can be expressed analytically only by means of transcendental equations.

In the seventh article the simplest type of problems of the first class is considered, that which corresponds to a cubic equation. Practically the whole chapter is devoted to the two simplest of such problems: that of *the doubling of the cube*, and that of *the trisection of an angle*—problems almost prehistoric in their origin. It is then shown that all cubic problems can be reduced to the solution of one of these two.

The proof of *the existence of transcendental problems* occupies the main portion of the eighth article. Some of the latest expositions of the fact that e and π cannot satisfy any algebraic equation are expounded at length.

---

*This leads me to remark that the usefulness of the book would be increased if the references to original sources were fuller. Thus, for instance, on p. 164, the statement that certain numbers have been found not to be primes should be supplemented by references.

† W. H. Young, *L'enseignement Mathématique* (1909), pp. 102-104.

‡ H. W. Richmond, *Quart. Journ. of Math.* Vol. XXVI, p. 206.

One of the most interesting and important of the articles in the volume brings it to a conclusion. It consists of general remarks by Enriques on Geometrical Problems. As this article appears for the first time in the German translation, I propose to discuss it somewhat fully. The first section (The Practical Aim of Geometrical Investigation) might well be a little less condensed. It deals with a favourite maxim of the Göttingen professor who inspired the volume. This is that a mathematical entity—whether a geometrical form, or some other form, say, a system of algebraic equations, or a group of operations—is only to be considered as perfectly known when the mind has apprehended it as a whole, with all its leading properties systematically collected or tabulated. It is only then that information in a novel form about the entity may be as readily tested and grasped as, say, a new rider on the properties of a triangle. Thus an equation between x, y and z can hardly be said to be perfectly known to the analyst who merely manipulates its equation. He has but the vaguest ideas about the surfaces in three dimensions which it is capable of representing, and in particular the simplest such surface. Its plane sections, its curvature, singular points and lines and so forth have at best a disjointed and imperfect existence as analytical peculiarities in his mind. Yet these things are of the utmost value, not only as a *memoria technica*, but as suggesting both the possibilities and the restrictions involved in the nature of the mathematical form in question.

The making and handling of models of the surface, and diagrams of its sections, gives a grasp of the object of our investigation as a three-dimensional whole which cannot be superseded. Hence the practical value, even in abstract mathematics of diagrams and models, both rough and detailed.

Perfect knowledge of a mathematical entity involves one more point, the recognition of those properties which taken together are characteristic of that entity and of no other. As an illustration, the following set of properties are cited as characteristic of the dodecahedron :

1. The faces are pentagonal.
2. Three edges meet at each vertex.
3. The projection of the edges on to the plane of one of the faces consists of two concentric regular decagons, such that the difference of the radii of the two circumcircles is to the smaller radius as the greater to the less segment of the latter when divided by the so-called "golden section" (Euclid II. 11).

After elucidating the terms "definite" and "indefinite" problems, Enriques passes on to the question of the most "economical" solution, a question on which the earlier chapters of the book serve to throw light. What constitutes "economy" from this point of view? We have to take into account not only saving of thought, that is simplicity in the proof, but also saving of work. In this last respect there will be a difference according to the conditions in which we find ourselves. If we have at our disposal all the delicate instruments which modern technique places before the mathematical public, we shall choose a solution which will from the point of view of economy be quite different from that we should adopt if thrown on our own resources. And here it must be remarked that the instruments at the disposal of most English students, even at the universities, are of the most inadequate description, and that even such an invaluable instrument as the integraph, the use of which is exemplified in several places in the book before us (pp. 204, 219, 297 seq.), is but rarely accessible.

When thrown on our own resources the most economical solution will involve the use of the ruler only, with or without the T-square, simple instruments which can be made in a moment, if not ready to hand. It is not so easy to fabricate compasses accurately, but when, as is usually the case, these are to hand, the use of them will, as already pointed out, materially increase the accuracy of the result. This question of accuracy is one which holds an important place in the estimation of the economy of a solution, and here we have two aspects to consider, firstly, the theoretical accuracy of a solution, and secondly, the practical accuracy and the limits of the error introduced by the use of many straight lines and circles.

The literature of this part of the subject has received some interesting

additions since the publication of the present book in its various forms. I should like to mention the approximate squaring of the circle by G. Pierce in the *Bulletin* of the American Mathematical Society, XIII. (1907), pp. 166-167, where references will also be found. Pierce succeeds by the use of four straight lines and five circles in constructing a straight line whose length differs from that of a given semicircle by less than .0005 of its length. If we compare this solution with that given in the book before us (pp. 302-3), it will be remarked that the theoretical accuracy of the latter solution is greater than that of Pierce, but the limits of error are also greater. The construction in the book involves beside the drawing of five straight lines and three circles, one use of the T-square, the division of a straight line into five equal parts, and the drawing of a parallel to a given straight line through a given point. Thus, although the description of the process and the accompanying figure on p. 303 are very simple, and the theoretical accuracy surprising, probably a more accurate result would in practice be obtained by using the approximation

$$\pi = \sqrt{10},$$

an approximation eminently adapted to geometrical interpretation.

The question of the instruments to be employed is adopted further on as a mode of classifying geometrical problems. It is pointed out that this question of classification is different according as it relates to *problems already solved*, or to *those which await solution*. The mechanical point of view will dominate the former, the analytical the latter. The first question with regard to an unsolved problem is, is it soluble? and, if so, what methods of attack can at once be dismissed as useless? The data of the problem will in general supply us with equations which enable us to answer these questions and suggest the best methods of attack.

I should like to call the attention of advanced students to the suggestive footnote on p. 331 dealing with the theoretical possibility of plotting down a curve.

It only remains to say that the public is indebted to all those concerned in the production of this book, whether to the inspirer, the editor, the authors, the painstaking translator or the publisher. The volume is in every respect to be recommended to the reading of the mathematical public.

W. H. YOUNG.

Allgemeine Formen- und Invariantentheorie: I Band, **Binare Formen.** By W. F. MEYER. Leipzig. (Sammlung Schubert.) 1909.

This is the first volume of a work of an essentially elementary character. Its aim is to be thorough as well as introductory, and it succeeds. It relates to invariants and covariants of Binary Forms. The future second volume is to deal in the same simple and searching way with the fundamentals of invariant algebra in respect of ternary and higher forms. The title "Allgemeine Theorie," given to a book from the pen of an encyclopædist like Dr. W. F. Meyer, might naturally be misinterpreted by an English reader as implying an invitation to enquire within upon everything. But, as a matter of fact, a restrictive and not an extensive interpretation must be given to the word "Allgemeine." Enquire within rather for enlightenment as to the common groundwork of all theory of invariancy under linear transformations, and look elsewhere for advanced special developments. The quintic and sextic are hardly mentioned. The formal theory of irreducibility, and investigations as to complete linearly independent systems of concomitants, are absent. Only in two appendixes of no great length are symbolic methods touched upon.

The author is, as all students of Higher Algebra know, second to no man living in profundity of learning about invariant algebra. The history of the development of this branch of mathematics, and the details of the most recent researches upon it, are alike familiar to him as the multiplication table. In his preface he now propounds the question how to account for the fact that intimate knowledge of a subject of such far-reaching importance to mathematicians is still not widely distributed among them. His answer to the question, weighty as coming from the best-informed authority, is one which must appeal to all who have pursued the study at all as strikingly in accord-

ance with their own vague impressions. "Bei der historischen Entwicklung der Invariantentheorie die verschiedenartigsten Methoden mitgewirkt haben," and the correlation of the variously expressed results of all these dissimilar methods is a serious task, which few find congenial. Even text-books differ radically both in procedure and in expression of conclusions. There is little but symbolism in one, and no symbolism at all in a second. A third presupposes much knowledge of geometry, and a fourth an intimate acquaintance with group-theory. A student whose interest has been awakened by one passes to another, and his interest is deadened by the discovery that he must begin again with drudgery in acquiring new sets of ideas, which eventually lead to results such as he can only with difficulty recognise as old friends in new guises. His knowledge, before one-sided, is confused more easily than extended.

If this view is correct, and it is to say the least plausible, the number of these who can hope to read with facility all the memoirs in which the various aspects of invariant theory were first regarded, and all the new literature on the subject which appears from year to year, must continue small. But there remains another question. Can the possession of the whole fortress, to outworks of which differently equipped forces have advanced by many routes, be secured by attack along a single line? Can one road of access be cleared of obstructions for a large company, and all the various outposts be made accessible from it? If so, let the history of campaigns be left as a study for the few. There is hope that a trade route will be established.

Dr. Meyer knows all the history of all the campaigns. He has moreover studied every detail of the map. Now he comes forward himself as a guide to the inexperienced who want to start along the best line of march. What path does he choose as the most promising? The question is interesting, and the answer one which gives the present critic personal satisfaction.

Enough of metaphor! The desideratum is to make the beginnings of invariant algebra simple and pleasing to those whose mathematical equipment is still small, and to secure that a real and tangible knowledge of the meaning and usefulness of the study shall result. The method adopted is direct and unencumbered by artificiality. The group notion as to linear substitutions is made fundamental, but the technicalities of group theory are not dwelt upon. The general linear substitution is analysed into a succession of one-parameter substitutions—translations and extensions—each of which provides a differential equation satisfied by an invariant form; and the facts as to degree and weight, and as to annihilation by those differential operators which play so great a part in the presentations of invariant theory to which English readers are most accustomed, are at once clearly exhibited, ever afterwards to be kept in sight.

Infinitesimal methods are rarely made use of, even when brevity could have been secured by their introduction. Indeed constant effort has clearly been made to resist temptation to save space, by appeal to analysis of which those young in algebra could only avail themselves with imperfect appreciation of assumptions tacitly made. Neither, on the few occasions when questions akin to that of convergency of expansions have to be reckoned with, is there any sparing of the prolongation of argument needed in order to combine precision with elementariness. Dr. Meyer is as patient in full expression when writing an elementary text-book as he is ruthless in condensation when producing an article for the *Encyclopädie*.

Though the book is one on the principles of general theory, there is a great deal of the particular in it, but of the particular on early and not on advanced matters. In fact, nearly half of it is devoted to a full consideration of quadratic and bilinear forms, an extended treatment of which in advance seemed to the author both important in itself, and well calculated to make more natural the general considerations to follow, by first exhibiting them in cases of great simplicity. Some may think that, in this desirable practice of leading up to the general from the easily understood particular, the author has gone to an extreme. The separation of invariant theory proper from considerations of algebraical manipulation and of geometrical illustration may have been too long deferred.

Perhaps the parts of the book where the English reader not new to the

subject will find himself on least familiar ground, and will discover that he has most to learn, are those in which non-cogredient substitutions are applied to the different sets of variables in forms which contain more sets than one—for instance, § 15 and § 17. The extension of the notion of fundamental seminvariants (protomorphs) to the case of such composite forms ought greatly to interest him.

If the book had been divided into shorter sections it would have been rather easier to read. The average length of a section is now 16 pages, and there is naturally often, in the course of one, a passage from one train of ideas to another, which might with advantage have been marked by a pause and a re-opening. Moreover, as there is no index, a table of contents with only 24 entries—well chosen as these are—is hardly an adequate help to one who would refer to the book for a detail of information.

The type of the Sammlung Schubert is large and clear, but the page is small, and lengthy analytical formulae do not find elegant expression upon it. The author has shown skill in reducing the introduction of such formulae to a minimum, but has thus perhaps been led to some small over-indulgence in brief notation and reference to numbered results.

Minor misprints are somewhat too numerous. The author has detected and pointed out many, but others remain. They do not, however, give the reader much trouble. E. B. ELLIOTT.

The Future of Mathematics. By PROF. G. A. MILLER. *Popular Science Monthly.* August, 1909. Pp. 117-123.

Prof. Miller here emphasises the economy of thought as one of the chief characteristics of mathematics, and then gives some of the main results of Poincaré's address on the future of mathematics, before the Fourth International Congress of Mathematicians, held at Rome in April, 1908. Poincaré's tenet that it is the duty of the mathematicians to discover and fix by a name a common property of different things is illustrated by an elementary example from the theory of groups. This duty is described, on p. 120, by the statement that "Mathematics is the art of giving the same name to different things"; but it seems that inaccuracy in expressing an inadequate thought is too great a price to pay for obtaining a not very profound paradox.

PHILIP E. B. JOURDAIN.

Elemente der Mathematik. M. JULES TANNERY, Professor an der Universität Paris, deutsche ausgabe von Dr. P. Klaess. Pp. ix, 339. 7 m. 1909. (Teubner.)

The object of this most interesting book is admirably stated in the preface by M. Tannery to the German edition, to the following effect.

In all countries there are plenty of young students who at the close of their attendance at a secondary school know a little arithmetic, a little algebra, a little geometry. And half of this knowledge is soon forgotten, for they do not perceive the coherence or the value of what they have learned. At a later period the doctor who wishes to follow up some question of arterial pressure or optical accommodation, the lawyer who occupies himself with economic questions, the traveller, the merchant, and above all, the technical student feel their lack of knowledge of mathematics. This defect is not irremediable, and it is the author's main object to show that only a little further progress is needed to gain a view of mathematics as a science one and indivisible. After an introduction and three preliminary chapters written for the class just specified by way of revision, the author deals with the outlines of analytical geometry, the notion of a curve and function, the notion of a tangent and differentiation, the notion of an area and integration. To quote again from the preface: "These are big words, with which great ideas are associated. Yes! But these ideas are in their essence simple."

A point of some interest in the present unsettled condition of geometry in England is the treatment of the theorem of Pythagoras by similar triangles. Besides the usual Euclidean proof, our modern manuals give two other types—a proof by dissection and a proof by similar triangles. The proof by dissection, if it is not to be a mere appeal to the eye, and therefore in no sense a proof, however convincing, involves somewhat tedious verifications of the

congruency of certain parts. Some very experienced authorities object to the proof by similar triangles, not as lacking in cogency, but because they assert that the student, when he first learns Euclid I. 47, is not in a position to appreciate the force of the argument from similarity. This is a weighty objection. An extreme case of the same kind was given years ago by Dr. Todhunter, who asked whether a small boy was entitled to prove the "Asses Bridge" by observing that the figure is symmetrical?

The question is one of fact: which can only be answered by the experience of teachers.

The theorem of Pythagoras is one of the theorems of elementary geometry which every student would admit requires proof. It is not obvious. Is the proof by similarity cheerfully accepted by beginners or not? * C. S. J.

Notes on Dynamics. By Sir George Greenhill. 221 pp. fol. 3s. net. 1908. (H.M. Stationery Office.)

This book may be described as the pure spirit of dynamics: from which each of us can mix his dynamical grog to taste. In the conventional phrase, but in no conventional sense, every teacher of mechanics in the empire should possess a copy, in the first place, for the amazing number of illustrative examples it contains. Innumerable incidents of life, from the steamer crossing the Atlantic and always making its record on the westward run, and from the goods train rumbling through suburban stations past the stopping local, which repasses it again, down to a pencil rolling off a desk, all contribute dynamical problems for the reader's pleasure. In the second place, for the manner in which geometry and analysis are employed side by side, as in the discussion of pendulum reactions at p. 169, or of the elliptic trajectory on p. 118; and in the third place, for the appearance (for the first time it is believed in an English text-book) of the principle (formulated by Sir George Greenhill himself in the "Annals of Mathematics" in 1904) that if OH represents the angular momentum of a dynamical system, the velocity of H represents the impressed couple, and is in the direction of the axis of that couple. How delightfully the discussion of the motion of a top is simplified and rendered intuitive by the use of this principle is far from being fully realised yet.

Sir George Greenhill is famous as a pure mathematician—witness his work on the elliptic functions; he is equally famous as an expert in ballistics and hydrodynamics; but many of us will believe that the writing of the "Treatise on Hydrostatics" and "Notes on Dynamics" is not the least of the services he has rendered to the science to which his life has been devoted. A. Q. H.

An Elementary Treatise on Spinning Tops and the Theory of Gyroscopic Motion. By H. Crabtree, Assistant Master at Charterhouse. Pp. xii, 140. 5s. 6d. net. 1909. (Longmans.)

The motion of a top is the typical example in applied mathematics of the power of analysis to transcend the utmost stretch of intuition.

Mr. Crabtree has set himself the problem of describing and analysing gyroscopic phenomena so that they may be understood and not merely accepted; and he has attained a greater measure of success than any previous writer known to the present reviewer.

If we try to specify why the motion of a top appears so paradoxical to our common sense, however firm our faith in the dynamical equations, we may perhaps say that whereas we are accustomed to regard rotation about an axis as due to forces having a moment round that axis we seem to find, in the case of a top, something causing rotation about an axis about which the external forces have no moment.

Mr. Crabtree brings out very clearly that this is mere seeming. The only way to alter angular momentum about an axis fixed in space is to apply forces having a moment about that axis.

The next step is to realise that angular momentum is a vector quantity which can be resolved in any direction. Why does a gyroscope with its axis hori-

---

* May we urge members of the Association to discuss this question?

zontal revolve round the vertical in the familiar experiment instead of falling down?

It cannot fall down without requiring a large angular momentum (the component of the angular momentum of the spin) about the vertical, and no torque is available to give it that momentum. So it cannot fall down. But why does it precess round the vertical?

The torque due to the weight produces at starting a slight rotation about a horizontal axis fixed in space. But this rotation involves, as already stated, a component angular momentum about the vertical axis: and there can be no angular momentum about the vertical axis.

So the particles settle it among themselves, and the gyroscope precesses at such a rate and in such a way, that the total angular momentum about the vertical remains zero. In an appendix (p. 134) an indication is given of how the particles settle it among themselves: this also forming a novel and valuable feature of the book.

The scheme of Mr. Crabtree's book is as follows: Some of the most striking phenomena of spinning objects are stated, as physical facts to be observed. The first principles of rotational dynamics are then developed, and applied in chapter III. to obtain the equation

$$I\,\Omega\omega = \text{Torque},$$

whereby the phenomena of uniform precession are explained.

The practical applications of rotating flywheels to the steering of torpedoes, the Brennan monorail, and Schlick's gyroscopic steadying of ships are dealt with in chapter V.

The first 78 pages, constituting the portion just described, are within the grasp of any careful reader with a slight knowledge of the main principles of mechanics. The remaining chapters, VI.-IX., dealing with the steady motion of the symmetrical top and with moving axes, make greater demands on the reader.

Reference must not be omitted to a graceful sonnet, which will be read with sympathetic appreciation by all Cambridge men, dedicating the book to Dr. Besant.

A few slight suggestions may be submitted for the author's consideration, in view of a second edition. At p. 36 does it *follow* that angular momentum obeys the parallelogram law? On page 60 there are two paragraphs. The second gives a most lucid explanation of the reason for precession; and might not the first paragraph be left out? Inertia causing a wheel to dip downwards may be justified by good authority, but so much activity in inertia seems itself to require explanation.

Owing to a familiar optical illusion, by which a rounded curve may be imagined as having its concavity either way, the indication of sense of rotation in several of the diagrams is not free from ambiguity.

Everyone interested in the phenomena of spinning bodies should make a point of studying this book, which constitutes a distinct advance towards the goal which our descendants, if not ourselves, will reach of finding the motion of a top intuitively clear.

C. S. J.

Examples in Elementary Mechanics. By W. J. Dobbs. Pp. xii, 344. 5s. 1908. (Methuen & Co.)

The special feature of this book is the attention given to simple quantitative experiments. Many members of the Mathematical Association will recollect that Mr. Dobbs gave a demonstration, with apparatus designed by himself, at the annual meeting in 1907. Everyone present must have been impressed by the simplicity of the means employed, which yet in the right hands give excellent results.

The pious founder has always had a weakness for bricks and mortar. Educational authorities are torn between love of inspectors and a passion for expensive apparatus. Mr. Dobbs does a special service to the cause of education at the present time by impressing on us the positive value of simplicity and directness as well as the negative merit of economy. Interspersed with the examples are many useful practical hints, from which there are few teachers who will not derive some benefit. One little slip

occurs at p. 23 and elsewhere. Lamé is an honoured name among mathematicians : but not as that of the proprietor of Lami's theorem. C. S. J.

Cassell's Elementary Geometry. By W. A. Knight, M.A., B.Sc. Pp. 253.

The ground covered by this book is about that of the first six books of Euclid. The arrangement is, first, definitions, then Experimental Geometry (which the author has given in reasonable space—only really useful constructions being given), and then Theoretical Geometry. Euclid's first 32 propositions appear in this order—13-15, 28-30, 32, 4, 5, 8, 26, 6, 18-20, 24-25, and the Problems. Playfair's Axiom is used, and, throughout the book, where a different proof of a proposition is given, Euclid's proof is added as an alternative.

The second book of Euclid is treated as geometrical illustrations of algebra. Many of the less interesting of Euclid's propositions in the 3rd and 6th books disappear, and in their place we get others, such as the concurrence of the medians of a triangle, etc., etc.

The author is careful to explain fully the really important ideas, such as congruent triangles, tangents, symmetry, etc., and the explanations are lucid. The treatment of ratio is confined to commensurable ratios. There is a good selection of examples, but not too many. Most of them are necessarily old friends, but there are a few new faces too, and Ptolemy's Theorem is relegated to a position among the examples.

The author has managed his task of compressing Euclid into one manageable text-book very well.

The Elements of Geometry in Theory and Practice, Parts I.-III. By A. E. Pierpoint, B.Sc. Pp. 387, including answers and indices. 3s. (Longman.)

This book is most elaborately got up, and provided with 345 figures in white lines on a black ground. It contains the substance of the first four books of Euclid, with a little about Scales, Graphs, Field Book, and other small matters. The sequence of the propositions of Euclid I. very nearly agrees with that of Mr. Knight, and is designed to suit the schedules at Oxford, Cambridge, and London.

The book is well written, and thoroughly modern in every way. Each part contains an Experimental, a Theoretical, and a Practical section in that order. The Experimental sections, which are a great feature in the book, lead the pupil by measurement to discover the truth of the theorems before he is given their formal proofs in the Theoretical sections. There are no less than 373 "experiments" to be done. The examples are over 1200 in number, and Part I. contains 30 sets of "drill" questions as well. These statistics will show that the method of the book is on the grand scale. One cannot help feeling that for the ordinary pupil the book is over elaborate, but to the teacher it should be a mine of ideas and examples, provided as it is with almost every kind of question it is possible to conceive—some of course very easy, but some quite fresh. To those teachers—and there are many—who do not take kindly to the "new geometry" the book should appeal very strongly.

W. M. Roberts.

Elementary Mechanics. By C. M. Jessop and T. H. Havelock. Pp. vii, 277. 4s. 6d. 1909. (Bell & Sons.)

This volume is a revised form of Mr. Jessop's "Elements of Applied Mathematics," omitting the portions of the earlier work dealing with Hydrostatics. There are useful additions in the shape of sections on bending moment, harmonic motion, and shearing force, with a new chapter on the energy of rotating bodies. We notice that many examples have been added to the various chapters. But throughout the book there is little or no sign that the movement for reform in teaching the subject has penetrated so far north as Durham. This will not, however, prevent the book being useful in the hands of a good teacher in charge of an experimental course. Indeed it is possible that it may be a very good exercise to set a class of sharp boys to detect the unrealities in a volume that a few years ago was considered an

extremely good specimen of its class. To put the case against all books of this type in the present stage of reformed teaching, we cannot do better than quote part of a sentence from an unsigned review in our contemporary, "Nature." It brings back to memory days of long ago, with its "forces of 1, $\sqrt{2}$, and $\sqrt{3}$ lbs., its Roman and Danish steelyard, its three classes of lever, the oar being included in the second regardless of the man's thrust on the rowlock, its mechanical advantage instead of the more modern velocity ratio and efficiency, its systems of pulleys which only lift a weight a small fraction of the height of the supporting beam, and perhaps do not lift it at all if the ropes are extensible, . . ."

A First Course in Analytical Geometry, Plane and Solid, with numerous Examples. By C. N. Schmall. Pp. 7+318. 6s. net. (Blackie.)

There is so little to differentiate Mr. Schmall's book from the ordinary text-book dealing with this subject that it is somewhat difficult to see what can have induced the publishers to import from the States a volume that appeared some four years ago. The one feature worthy of note and, we may add, of imitation, is the early introduction and general application throughout of the determinant notation. The chapter on Higher Plane Curves covers but 17 pages, and is too scrappy to be of much use, while the point, plane, straight line, and surfaces of revolution are dismissed in some three dozen pages. Paragraphs 180 and 181 seem to be out of their place in the chapter on Confocal Conics, and the x^2 has disappeared from equation (1) on p. 227. The book is well printed and illustrated, and the examples for solution are numerous and seem to be well chosen. The treatment is sound, and the exposition is clear. The book must have some merits we have not been able to discover if it is to compete with its rivals at the price of 6s. net, at which it is issued.

A Brief Course in the Calculus. By W. Cain. Pp. x+280. 6s. net. 1909. (Blackie.)

Mr. Cain's "Brief Course" was also published in 1905, and comes to us from the States, but with the additional credential of having attained the dignity of a second edition. The first forty pages are devoted to coordinate geometry, and towards the end of the book we have fifty given to the integral calculus so that the differential calculus takes up about two-thirds of the whole space. Although the book is ostensibly intended for those who have a "thorough knowledge of the elements of Trigonometry," we find a page of formulae at the end of the volume containing the value of $\sin^2\alpha+\cos^2\alpha$, $\sin(\alpha+\beta)$, $\tan(90+\theta)$ and the like. The "working knowledge of Algebra" the student is supposed to possess apparently does not extend to the use of factorials, which have to be defined in a footnote. The ideal the author has set before himself is to provide material so that the reader may "hammer away at the essentials until they are thoroughly mastered." Within the limits he has set himself Mr. Cain may be fairly said to have achieved his purpose. But a six shilling (net) hammer is rather an expensive tool for boys who have to be reminded that

$$\cos^2\alpha+\sin^2\alpha=1, \text{ and that } \underline{|n}=n(n-1)(n-2)\ldots 3.2.1.$$

Wahrscheinlichkeitsrechnung, und ihre Anwendung auf Fehlerausgleichung Statistik und Lebensversicherung. By E. Czuber. Vol. I. **Wahrscheinlichkeitstheorie; Fehlerausgleichung; Kollektivmasslehre.** Second enlarged and carefully revised edition. Pp. x, 410. (Teubner.)

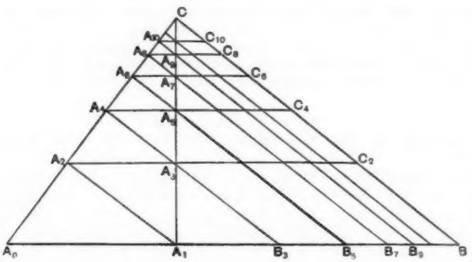
The first volume of the first edition, published in 1902, contained about 300 pages, whereas the book before us contains over 400, so that the author might have fairly described this edition as "considerably" enlarged. The chapter on geometrical probabilities is increased by twelve pages, that on Bernoulli's Theorem by eight, and so on, while there is an entirely new chapter of 35 pages on the youngest of the allied branches of the subject, Kollektivmasslehre. Here the author has availed himself of the important results achieved by Messrs. Lipps and Brun, whose researches were published in 1902 and 1906 respectively, and therefore before Dr. Czuber could avail himself of them. The additions will

greatly enhance the value of the book to the student and to all professional statisticians. Dr. Czuber's volume will supply to teachers of what we call higher algebra much that will be interesting and amusing to their best pupils. There is something in probabilities that appeals to boys—they appear to afford more of a human than a purely intellectual interest, at any rate in the early stages of the subject.

MATHEMATICAL NOTES.

291. [X. 4. a.] *A Geometrical Representation of the Sum of an Infinite Geometric Series.*

In the figure A_0A_1C is a right-angled triangle having its sides in the ratio 3 : 4 : 5. Perpendiculars are drawn in succession on the sides A_0C and A_1C of this triangle, commencing with A_1. The lengths A_0A_1, A_1A_2, A_2A_3, etc., represent the geometric series 3, $\frac{4}{5}.3$, $\left(\frac{4}{5}\right)^2 3$, etc. If, now, we draw through C a perpendicular to A_0C to meet A_0A_1 produced at B, and produce A_2A_3, A_4A_3, etc., to meet the opposite sides, the odd terms of this series are



given by A_0A_1, A_1B_3, B_3B_5, B_5B_7, etc., and the even terms by BC_2, C_2C_4, C_4C_6, etc.

The sum of the infinite series

$$3+\frac{4}{5}.3+\left(\frac{4}{5}\right)^2.3+\left(\frac{4}{5}\right)^3.3+\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

will therefore be represented by A_0B+BC.

The length of A_0B is $8\frac{1}{3}$ and the length of BC is $6\frac{2}{3}$.

The sum of the infinite series is then 15.

Also the sum of the infinite series

$$3-\left(\frac{4}{5}\right).3+\left(\frac{4}{5}\right)^2.3-\left(\frac{4}{5}\right)^3.3+\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

is represented by $A_0B-BC=8\frac{1}{3}-6\frac{2}{3}=1\frac{2}{3}$.

The sum of an odd number of terms (say 9) of (i) will be $A_0B_9+B_9A_9$, and of (ii) $A_0B_9-B_9A_9$; and the sum of an even number of terms (say 10) will be (i) $A_0B_9+B_9A_{10}$ and (ii) $A_0B_9-B_9A_{10}$.

By taking the sums in the reverse order we get a geometric series in which the common ratio is greater than unity.

Obviously any geometric series may be represented in this way. The angle A_0CA_1 is drawn so that $\cos(A_0CA_1)$=the common ratio, or if the common ratio is greater than unity, so that $\sec(A_0CA_1)$=the common ratio. Then A_0A_1 is made equal to the first term.

R. M. Milne.

292. [X. 4. a.] *A Geometrical Construction for the sum of a Geometrical Progression.*

I have never seen any reason given for the name Geometrical Progression and do not know the author of the name. Euclid, who talks of continued proportion, uses a method which is essentially arithmetical when he finds the rule for the sum of a finite number of terms of a progression in the form: "If as many numbers as we please be in continued proportion and

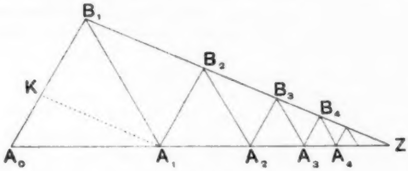


FIG. 1.

there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it." Euc. IX. 35.

It is not difficult to find geometrical representations of the progression. Perhaps the simplest is the following.

Along a straight line mark lengths A_0A_1, A_1A_2, A_2A_3 ... to represent the terms of the progression. On these as bases describe equilateral triangles with their vertices at B_1, B_2, B_3, ... Since the angles $B_1A_1B_2$ and $B_2A_2B_3$ are equal, and the sides containing these angles are proportional, the triangles $B_1A_1B_2$ and $B_2A_2B_3$ are equiangular, and hence B_1, B_2, B_3 ... are collinear. By increasing the number of terms, we can make the triangle B_nA_nZ as small as we please, Z being the point when B_1B_2 cuts A_1A_2. Therefore the "sum to infinity" of the progression is A_0Z.

To calculate the sum, draw A_1K parallel to B_2B_1 and meeting A_0B_1 at K. By similar triangles,

$$A_0Z : A_0B_1 = A_0A_1 : A_0K,$$

or, in the ordinary notation,

$$S : a = a : a - ar,$$

$$S = \frac{a}{1-r}.$$

When r is negative the same method can be used, the equilateral triangle corresponding to a negative term being inverted (see Fig. 2).

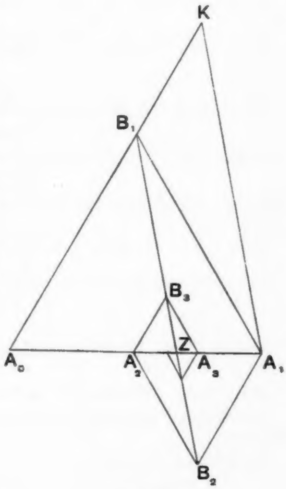


FIG. 2.

The most valuable example for bringing home the meaning of the sum to infinity is perhaps that of a bouncing ball: a purely geometrical question is found in the trisection of an angle by the use of the identity

$$\tfrac{1}{3} = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} \dots .$$

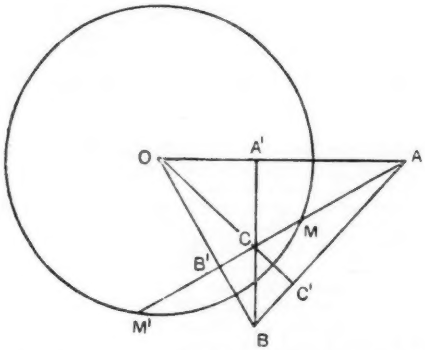
F. J. W. WHIPPLE.

293. [**K. 10. b.**] *Simple proof of the harmonic property of pole and polar of a circle.*

For definition of pole and polar with reference to a circle we choose the elementary one, that if O be the centre and O, P, P' a range of points such that $OP.OP'=r^2$, then the line through P' perpendicular to OPP' is the polar of P.

Let O be the centre of a circle, A any point, and AMM' any line through A cutting the circumference in M and M' and the polar of A in C.

Complete the triangle OAC and draw the perpendiculars AC', OB', CA' from its angular points to the opposite sides, and let them meet in the orthocentre B of the triangle.



Since A, A', B', B lie on the circle whose diameter is AB, we have

$$OB.OB'=OA.OA'=r^2,$$

and since A, A', C, C' lie on the circle whose diameter is AC,

$$\therefore\ OC.OC'=OA.OA'=r^2.$$

(Hence each side of the triangle ABC is the polar of the opposite angular point.)

Now
$$\begin{aligned} B'C.B'A &= B'C^2+B'C.CA \\ &= B'C^2+OC.CC', \text{ since } O, B', C', A \text{ are concyclic,} \\ &= OC^2-OB'^2+OC.CC' \\ &= OC.OC'-OB'^2 \\ &= r^2-OB'^2=B'M^2. \end{aligned}$$

Hence C and A are harmonic conjugates with reference to M, M'.

SOLIDUS.

294. [**V. a.**] *Tangents to conics.*

In teaching students who have begun the Calculus I use the following methods:

(1) *Bifocal properties.*

Ellipse.	Hyperbola.	Parabola.
$S_1P+S_2P=\text{const.},$	$S_1P-S_2P=\text{const.},$	$SP=MP,$
or $r_1+r_2=\text{const.},$	$r_1-r_2=\text{const.},$	$r=x;$
$\therefore\ \frac{dr_1}{ds}+\frac{dr_2}{ds}=0,$	$\frac{dr_1}{ds}-\frac{dr_2}{ds}=0,$	$\frac{dr}{ds}=\frac{dx}{ds};$
$\therefore\ \cos\phi_1+\cos\phi_2=0,$	$\cos\phi_1-\cos\phi_2=0,$	$\cos\phi=\cos\psi;$
$\therefore\ \phi_1+\phi_2=180°,$	$\phi_1=\phi_2,$	$\phi=\psi;$

$\therefore$ tangent bisects exterior angle between focal radii.

tangent bisects interior angle between focal radii.

tangent bisects angle between SP and $\perp$ on directrix.

(2) *Normal property.*

$$SP=eMP\,; \quad \therefore\ r=ex\,; \quad \therefore\ \frac{dr}{ds}=e\frac{dx}{ds}\,; \quad \therefore\ \cos\phi=e\,.\cos\psi,$$

giving $\sin SPG=e\,.\sin SGP$ or $SG=e\,.SP$.

The following mechanical proofs are also useful:

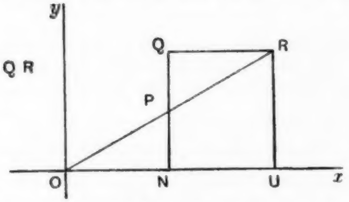
(1) *Bifocal properties.*—Let a particle constrained to move on an ellipse be pulled to the foci by equal forces F. Then since $S_1P+SP_2=$const., if the particle move along the ellipse, the increase in S_1P is equal to the decrease in S_2P; therefore the sum of the works done by the forces is zero; therefore the particle is in equilibrium; therefore the normal is in the direction of the resultant of the pulls F, and bisects the angle S_1PS_2. For the hyperbola we must take one of the forces attractive and the other repulsive.

(2) *Normal property.*—Let the conic be placed in a vertical plane with directrix horizontal and above the curve. Let a weight eW be placed on the curve and attached to a so-called inextensible string passing through the focus and supporting a hanging weight eW. Then since when the weights move, the increase in SP is e times the increase in MP, the sum of the works done is zero, and the weights are in equilibrium. Moreover, the triangle SPG is a triangle of forces for the weight eW at P, giving at once $SG=e\,.SP$.

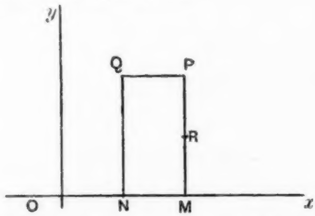
G. H. BRYAN.

295. [M[1]. 8. g.] *Graphic construction of the parabolic and hyperbolic curves* $y=x^n$.

The curves for positive or negative integral values of n may be derived in



succession from the graph of $y=1$ by a simple construction for the graphs of $xf(x)$, $1/x\,.f(x)$ when that of $f(x)$ is given.



Let Q be any point on $y=f(x)$. Draw a parallel to Ox to cut the line $x=1$ in R. Then the meet P of OR with the ordinate NQ of Q is a point on the graph of $y=xf(x)$.

For $NP/ON=UR/OU$; *i.e.* $NP=ONf(ON)$.

Conversely, if P is a point on $y=f(x)$, we get a point Q on $y=1/x\,.\,f(x)$ by drawing through the meet R of OP with the line $x=1$ a parallel to Ox, cutting the ordinate NP of P in Q.

The curves for positive or negative fractional values of n may be derived from those for integral values of n.

Let any parallel to Ox cut the curves $y=x^p$, $y=x^q$ in P, Q. In MP take R so that $MR=ON$, then R is a point on $y=x^{\frac{p}{q}}$.

E. J. NANSON.

296. [K. 21. d.] *Approximate construction for a radian.*

The following perversion of Rankine's (I believe) rectification of the arc of a circle leads to a very simple formula and construction for a radian.

To construct an arc equal to the radius of a circle.

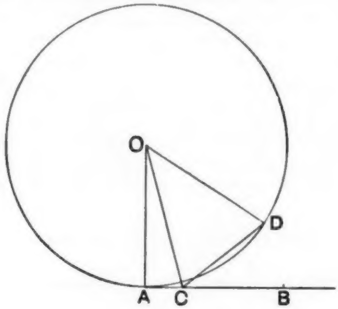


FIG. 1.

Take any point A on the circle, and on the tangent at A, take B such that $AB=$radius, and C such that $AC=\frac{1}{4}AB$. With C as centre and CB as radius, describe arc cutting the circle in D. The arc AD is approximately equal radius.

Now let O be the centre of the circle.

Join $OAOC$, CD. Then if $OA=4$, $AC=1$, $OC=\sqrt{17}$, $CD=3$, $OD=4$;

$$\therefore \quad \cos COD=\frac{17+16-9}{2\,.\,4\,.\sqrt{17}}=\frac{3}{\sqrt{17}};$$

$$\therefore \quad \sin COD=\frac{2\sqrt{2}}{\sqrt{17}};$$

$$\therefore \quad \tan COD=\frac{2\sqrt{2}}{3},$$

$$\text{and} \quad \tan COA=\frac{1}{4};$$

$$\therefore \quad \tan AOD=\frac{\frac{1}{4}+\frac{2\sqrt{2}}{3}}{1-\frac{2\sqrt{2}}{12}}=\frac{68+102\sqrt{2}}{136}$$

$$=\frac{2+3\sqrt{2}}{4};$$

$$\therefore \quad AOD=\tan^{-1}\frac{2+3\sqrt{2}}{4}=\tan^{-1}1\cdot560660=57^\circ\,21'\,0'',$$

$$1 \text{ radian}=57^\circ\,17'\,45''.$$

Error is $3\frac{1}{4}$ minutes, or less than $\frac{1}{100}''$ of arc on a 10 inch circle;
i.e. exact for all purposes of drawing.

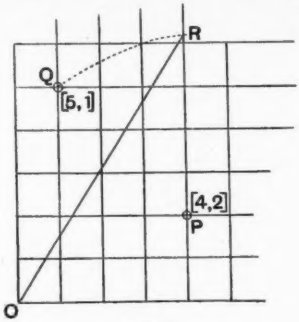


Fig. 2.

This angle is easily constructed on accurate "squared paper" as shewn in Fig. 2. J. M. Child.

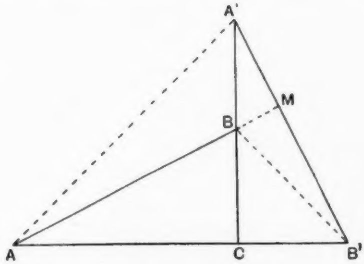
297. [K. 3. c.] Euc. I. 47.

Let ABC be a triangle right-angled at C.

Then must $$CA^2+CB^2=AB^2.$$

Let $\triangle ABC$ be turned about C through $90°$ to the position $A'CB'$.

Then A', B, C are collinear, B', C, A are collinear, and $A'B'$ is perpendicular to AB.



Produce AB to meet $A'B'$ in M; join AA', BB'.

$$\triangle ACA'+\triangle BCB'=\triangle ABA'+\triangle ABB';$$
$$\therefore\ \tfrac{1}{2}CA\,.\,CA'+\tfrac{1}{2}CB\,.\,CB'=\tfrac{1}{2}AB\,.\,A'M+\tfrac{1}{2}AB\,.\,MB'$$
$$=\tfrac{1}{2}AB(A'M+MB')=\tfrac{1}{2}AB\,.\,A'B';$$
$$\therefore\ CA\,.\,CA'+CB\,.\,CB'=AB\,.\,A'B';$$
$$i.e.\ CA^2+CB^2=AB^2.$$

Cecil Hawkins.

298. [K. 5.] *Orthologic triangles.*

Let the triangles ABC, LMN be orthologic, so that the perpendiculars from A, B, C on MN, NL, LM meet in P, while the perpendiculars from L, M, N meet in Q.

To prove that P, Q have proportional barycentric coordinates: P in ABC, Q in LMN.

Since PA is perpendicular to MN, the sides of the anti-pedal triangle of P in ABC are parallel to the sides of LMN.

Hence P is the point whose anti-pedal triangle has the angles L, M, N.

P is the common point of intersection of circular arcs on BC, CA, AB drawn inwards, and containing angles $\pi - L$, $\pi - M$, $\pi - N$.

Its barycentric coordinates are found at once to be as $\dfrac{\sin A \sin L}{\sin(A+L)}$, etc.

The symmetry of these expressions proves the theorem.

For given angles L, M, N there is only *one* point P, all the triangles LMN being homothetic.

W. Gallatly.

NOTICE.

MATHEMATICAL ASSOCIATION LIBRARY.

Wanted by purchase or exchange :

1 or 2 copies of *Gazette* No. 2 (very important).
1 copy ,, No. 3.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 ,, ,, No. 10, 12 (very important).
1 copy ,, No. 1, 2.

QUERY.

(67) C is the centre and F, F' the foci of an inconic of a $\triangle ABC$ whose circumcentre is S, circumradius R, and N.P. centre N.

Prove that $SF \cdot SF' = 2R \cdot CN$.

Sonti V. Ramamurty.

CORRESPONDENCE.

South Kensington, S.W.,
4*th August*, 1909.

Sir,—In the issue of the *Mathematical Gazette* for June–July, 1909 (page 93), the following passage relating to the University of London occurs in an article entitled "An Arbitrary Veto":

"For a teacher to be recognized by the University (and many posts are granted but on this condition) the teacher must not deal with any subject other than the one in which he wishes to be recognized. Before anything can be done this regulation must be abolished."

I desire to point out that no such Regulation exists. I may add, in this connexion, that Section 88 of the University of London Statutes reads as follows:

"In appointing or recognising a Teacher of the University the Senate shall specify the subject, that is to say the branch or branches of knowledge for which he is appointed or recognised, and shall take care that no Teacher is appointed or recognised for two or more branches of knowledge, unless those branches are of such a kind that in the opinion of the Senate instruction in them of a University standard can with advantage be given by the same person."

Under the terms of the Statute, recognition in two or more subjects is accorded to a particular teacher when the Senate deem such a course desirable. I am, Sir, yours faithfully, P. J. Hartog, *Academic Registrar.*

ERRATA AND ADDENDA.

p. 95, III. (1) For $\Sigma\lambda \cdot BC = 0$ read $\Sigma\lambda \cdot BC = 2\Delta$.
p. 98, VI. After "OP_1P_2 are collinear" add "and P_1 bisects OP_2."
p. 104, l. 5 up. For "continue" read "continua."
p. 104, l. 4 up. Close inverted commas after "not out," not after "world."

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.